AMS Common Exam - Part A, January 2017

Name: ________________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show that $\text{Ker}(\mathbf{A}) \perp \text{Im}(\mathbf{A}^T)$.

2. Let $\mathbf{M} = \begin{bmatrix} 5 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Is $\mathbf{M}$ invertible? Is $\mathbf{M}$ defective? Justify your answers.

(b) Determine a basis for each 1-dimensional invariant subspace of $\mathbf{M}$.

3. Let $S = \{e_1, e_2, \ldots, e_r\}$ be an orthonormal set of vectors in V $(r \le \dim(V))$, let $v \in V$, and let $c_k = \langle v, e_k \rangle$ for $k = 1, 2, \ldots, r$. Show that $\sum_{k=1}^{r} c_k^2 \le \|v\|^2$.

4. Let $S = \{u_1, u_2\}$ be a basis for V and let $T : V \to V$ be linear. Suppose $R = \{w_1, w_2\}$ is also a basis for V with $w_1 = u_1 + u_2$ and $w_2 = 2u_1 + 3u_2$. Determine $[T]_R$ if

$$[T]_S = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}.$$

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Show that $\dfrac{1}{2e} \leq \displaystyle\sum_{n=1}^{\infty} ne^{-n^2} \leq \dfrac{3}{2e}$.

2. Determine the minimum and maximum values of $f(x, y) = 2x^2 + y^2 - y + 10$ subject to $x^2 + y^2 \leq 1$.

3. Let $a > 0$. A cylindrical hole of radius a is cut into a sphere of radius $2a$. The axis of the cylindrical hole passes through the center of the sphere. Find the volume of the resulting bead (sphere with hole).

4. Suppose C is a simple, closed, positively-oriented curve in the xy-plane.

(a) Determine the curve C that maximizes $\oint_C \big(y^3 \mathrm{d}x + (12x - x^3)\mathrm{d}y\big)$.

(b) What is the maximum value of the integral?